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q-Deformation of the Double Complex Ernst Equation

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From a known three-dimensional representation of the double quantum algebra $su_q(\eta(J))$, a nonlinear q-deformed Ernst equation system is obtained. By using a gauge covariant form, the deformation effects are found to generate a torsion in the field and to form a gauge field with source.

KEY WORDS: double quantum algebra $su_q(\eta(J))$; q-deformed Ernst equation; deformation effects.

1. INTRODUCTION

In the past 10 years or so quantum group (Drinfel'd 1985; Drinfeld, 1986; Kulish and Reshetikhin, 1983) have been found to be important in many branches of physics (Belavin *et al.*, 1984; Knizhnik and Zamolodchikov, 1984; Verlinde, 1988). Many new realizations of the quantum group $SU_q(N)$, especially $SU_q(2)$, have been obtained. These results show that the properties of quantum groups are quite similar to those of classical Lie groups in the case of q not being a root of unity. It has been shown that rotational spectra of nuclei and molecules can be described very accurately in terms of a Hamiltonian that is proportional to the Casimir operator of the quantum group $SU_q(2)$ (Bonatsos *et al.*, 1991; Raychev *et al.*, 1990). But the applications of the quantum groups to classical field theory are few. Zhong (1992) discussed the double quantum algebra $su_q(\eta(J))$ that is believed to link with the gravitational field equations. Moreover, Feng and Zhong (1996) gave the q-deformed double complex Ernst equation that in fact describes the q-deformed gravitational wave field with cylindrical symmetry. In this paper, we get a nonlinear system that is a true q-deformed double complex Ernst equation.

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Furthermore, the physical meaning of the q-deformation is discussed. We find that the deformation effects are to generate a torsion in a proper gravitational field, to separate the gauge field into parts with distinct properties, and to form a gauge field with source.

The organization of this paper is as follows: in Section 2, the q-deformed double complex Ernst equation based on the quantum algebra $su_q(\eta(J))$ is given. In Section 3, by using the covariant form the physical and geometric explanation of the q-deformation effects is discussed. Section 4 gives the conclusion.

2. THE q-DEFORMED ERNST EQUATION

The general double complex function method has been discussed in several papers (Zhong, 1985, 1988, 1989). We only use the relevant results. Let J denote the double-imaginary unit, i.e., $J = i(i^2 = -1)$ or $J = \varepsilon(\varepsilon^2 = +1, \varepsilon \neq \pm 1)$.

According to Biedenharn (1989) and Zhong (1992), in the three-dimensional case the generating operators J_3^q and $J_\pm^q$ of the quantum algebra $su_q(\eta(J))$ can be represented as

$$J_3^q = J_3, \quad J_+^q = \left(\frac{\Delta}{2}\right)^{\frac{1}{2}} J_+, \quad J_-^q = \left(\frac{\Delta}{2}\right)^{\frac{1}{2}} J_- \tag{1}$$

where J_3 and $J_\pm$ are the generating operators of the $su(\eta(J))$ algebra with

$$J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & J^3 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix},$$

$$(\eta(J)) = \begin{pmatrix} 1 & 0 \\ 0 & +J^2 \end{pmatrix}, \quad \Delta = q + q^{-1} \tag{2}$$

The quantum number is

$$[\chi]_q \equiv \frac{q^\chi - q^{-\chi}}{q - q^{-1}} = \frac{\sinh(\gamma\chi)}{\sinh\gamma}, \quad \gamma = \ln q, \quad q \in (0, 1] \tag{3}$$

Let

$$L_1^q = \frac{J^3}{2} J_3^q, \quad L_2^q = \frac{1}{2J}(J_+^q + J_-^q), \quad L_3^q = \frac{1}{2}(J_+^q - J_-^q) \tag{4}$$

then the commutation relations are

$$\left[L_1^q, L_2^q\right] = \frac{J^2}{2}\Delta L_3^q, \quad \left[L_2^q, L_3^q\right] = -L_1^q, \quad \left[L_3^q, L_3^q\right] = \frac{\Delta}{2} L_2^q \tag{5}$$

Obviously, when $q \to 1$, L_i^q changes into the infinitesimal generating element $L_i (i = 1, 2, 3)$ of the Lie group $SU(\eta(J))$. This means that the three-dimensional representation of $su_q(\eta(J))$ is special, i.e., it is also a Lie algebra. In fact, the transformation form from L_i^q to $L_i (i = 1, 2, 3)$ is an affine isomorphism

$$L_1^q = L_1, \quad L_2^q = \left(\frac{\Delta}{2}\right)^{\frac{1}{2}} L_2, \quad L_3^q = \left(\frac{\Delta}{2}\right)^{\frac{1}{2}} L_3 \tag{6}$$

where

$$L_1 = \frac{J^3}{2} J_3, \quad L_2 = \frac{1}{2J}(J_+ + J_-), \quad L_3 = \frac{1}{2}(J_+ - J_-) \tag{7}$$

By using of the above results, we consider the following equation

$$F_{\mu\nu}^q = \partial_\mu M_\nu^q - \partial_\nu M_\mu^q - \left[M_\mu^q, M_\nu^q\right] = 0, \quad (\mu, \nu = 1, 2) \tag{8}$$

where

$$\begin{aligned}
&\partial_1 \varphi^q = M_1^q \varphi^q, \quad \partial_2 \varphi^q = M_2^q \varphi^q, \quad \varphi_k^q = \varphi_k^q(x; y; q, J) \ (k = 1, 2, 3),\\
&M_1^q = -\frac{\Delta}{2} \frac{\varepsilon_1^q - \bar{\varepsilon}_1^q}{\varepsilon^q + \bar{\varepsilon}^q} J_3^q + \frac{s\bar{\varepsilon}_1^q - c\bar{\varepsilon}_2^q}{\varepsilon^q + \bar{\varepsilon}^q} J_+^q + \frac{c\varepsilon_2^q - s\varepsilon_1^q}{\varepsilon^q + \bar{\varepsilon}^q} J_-^q,\\
&M_2^q = -\frac{\Delta}{2} \frac{\varepsilon_2^q - \bar{\varepsilon}_2^q}{\varepsilon^q + \bar{\varepsilon}^q} J_3^q + \frac{s\bar{\varepsilon}_2^q - c\bar{\varepsilon}_1^q}{\varepsilon^q + \bar{\varepsilon}^q} J_+^q + \frac{c\varepsilon_1^q - s\varepsilon_2^q}{\varepsilon^q + \bar{\varepsilon}^q} J_-^q
\end{aligned} \tag{9}$$

with the bar denoting complex conjugation as usual. $\varepsilon^q = f^q + J\psi^q$; $f^q(x, y; q)$ and $\psi^q(x, y; q)$ are real functions. s and c satisfy the condition

$$c^2 - J^2 s^2 = 1 \tag{10}$$

where

$$c = c(x, y) = \frac{1 \mp V^2}{1 \pm V^2}, \quad s = s(x, y) = \frac{2V}{1 \pm V^2} \tag{11}$$

and the real function $V = V(x, y)$ is a solution of the equations

$$(1 + V^2)\partial_1 V + \frac{1}{x}(1 \mp V^2)V = 0, \quad (1 \pm V^2)\partial_2 V - \frac{1}{2x}(1 \mp V^2)^2 = 0 \tag{12}$$

The concrete form of V can be easily written out. We see that, in fact, c and s are the common cosine and sine functions for $J = i$ and for $J = \varepsilon$, c, and s are the hyperbolic cosine and sine functions. From Eqs. (9)–(11), Eq. (8) changes into a double q-deformed Ernst equation

$$\begin{cases} \mathrm{Re}(\varepsilon^q)\nabla_{(J)}^2 \varepsilon^q = \frac{q+q^{-1}}{2} \nabla_{(J)}\varepsilon^q \cdot \nabla_{(J)}\varepsilon^q \\ \varepsilon^q = f^q + J\psi^q \end{cases} \tag{13}$$

where *Re* denotes the real part, and the operators

$$\nabla_{(J)}^2 = \partial_x^2 + J^2 \frac{1}{x} \partial_x + \partial_y^2, \quad \nabla_{(J)} = (\partial_x, J\partial_y) \tag{14}$$

When $q \to 1$, Eq. (13) becomes

$$\begin{cases} \mathrm{Re}(\varepsilon)\nabla^2_{(J)}\varepsilon = \nabla_{(J)}\varepsilon^q \cdot \nabla_{(J)}\varepsilon \\ \varepsilon = f + J\psi \end{cases} \tag{15}$$

This is just the double complex Ernst equation (Zhong, 1985, 1988, 1989). The case $J = i$ describes the axisymmetric gravitational field (Ernst, 1968), and the case $J = \varepsilon$ describes cylindrical gravitational waves (Letellier, 1984). Therefore, Eq. (13) represents a double quantum deformation of the common Ernst equation. We can regard ε^q as a q-deformed potential that describes the proper q-deformed gravitational field.

Although the difference between Eqs. (13) and (15) is only a coefficient $\frac{1}{2}(q + q^{-1})$, Eq. (13) is a completely new type of nonlinear differential equation. For example, if $\varepsilon^q = \exp(f^q)$ is a real function, then we obtain an elliptic type of quasi-linear equation

$$\nabla^2 f^q = \frac{1}{2}\left(\sqrt{q} - \sqrt{q^{-1}}\right)^2 \nabla f^q \cdot \nabla f^q \tag{16}$$

When $q = 1$, it returns to the Weyl solutions

$$\nabla^2 f = 0 \tag{17}$$

Of course, as a quantum group is not a group in the proper sense, it can not be guaranteed that f^q and ψ^q derived from M^q also satisfy the Einstein vacuum field equations. This means that the q-deformation result may not be a proper gravitational field.

3. THE COVARIANT FORM AND THE q-DEFORMATION EFFECTS

To explain what deformation effects are contained in ε^q, the results concerned must be written into a "covariant" form. However, it must be stressed that the "covariance" here is different from the common case. On one hand, it is not the covariance under an arbitrary coordinate transformation as in the general relativity. It is the pure gauge covariance with respect to a group G^q defined as in the following. And the new q-deformed solutions are generated by those gauge transformations. On the other hand, if we think (t, z) is a two-dimensional space-time with the signature $\eta_{\mu\nu} = \mathrm{diag}(1, J^2)$, and (M_1^q, M_2^q) is taken as a Lorentz covariant vector, then we may obtain the relativistic gauge field equations. But it is easily seen that the result of (M_1^q, M_2^q), under a Lorentz transformation, generally does not correspond to a new q-deformed solution ε^q. This means that even the Lorentz rotation of (t, z) is still not allowed. Therefore, in the following the coordinates are fixed as the cylindrical coordinates $(x, y, \tau, \sigma) = (t, z, \tau, \sigma)$. And (t, z) is regarded as an absolute two-dimensional space-time. Only for convenience we write

$$M_1^q = M_t^q = M^q, \quad M_2^q = M_z^q = N^q \tag{18}$$

For a localization gauge transformation

$$T_s : \varphi^q \to \tilde{\varphi}^q = S\varphi^q \quad S \in SU(\eta(J)) \tag{19}$$

M_μ^q ($\mu = 1, 2$) can be taken as the connection. Therefore, one can define the covariant derivative ∇_μ. For example

$$\nabla_\mu \varphi^q = \partial_\mu \varphi^q - M_\mu^q \varphi^q \tag{20}$$

The transformation rule of M_μ^q is

$$T_s : M_\mu^q \to \widetilde{M}_\mu^q = S M_\mu^q S^{-1} + (\partial_\mu S) S^{-1} \tag{21}$$

Therefore, from Eq. (18) one can obtain a covariant equation

$$\nabla_\mu \varphi^q = 0 \tag{22}$$

The gauge field

$$F_{\mu\nu}^q = \partial_\mu M_\nu^q - \partial_\nu M_\mu^q - \left[M_\mu^q, M_\nu^q\right] \tag{23}$$

is antisymmetric. It has only one independent component $F_{12}^q = -F_{21}^q$. The integrability condition asks for the covariant equation

$$F_{\mu\nu}^q = 0 \tag{24}$$

i.e., the gauge field vanishes. Of course, one always have

$$\nabla_\rho F'_{\mu\nu} = 0 \tag{25}$$

Now we discuss the concrete case of gravitational field that is more complex than the above. If ε^q is a q-deformed Ernst solution, then M_μ^q can be written as

$$M_\mu^q = \frac{\Delta}{2} m_\mu^{q1} J_3^q + m_\mu^{q2} J_+^q + m_\mu^{q3} J_-^q \tag{26}$$

where

$$m_1^{q_1} = -\frac{\varepsilon_1^q - \overline{\varepsilon}_1^q}{\varepsilon^q + \overline{\varepsilon}^q}, \quad m_1^{q_2} = -\frac{s\overline{\varepsilon}_1^q - c\overline{\varepsilon}_2^q}{\varepsilon^q + \overline{\varepsilon}^q}, \quad m_1^{q_3} = \frac{c\varepsilon_2^q - s\varepsilon_1^q}{\varepsilon^q + \overline{\varepsilon}^q},$$
$$m_2^{q_1} = -\frac{\varepsilon_2^q - \overline{\varepsilon}_2^q}{\varepsilon^q + \overline{\varepsilon}^q}, \quad m_2^{q_2} = -\frac{s\overline{\varepsilon}_2^{-q} - c\overline{\varepsilon}_1^q}{\varepsilon^q + \overline{\varepsilon}^q}, \quad m_2^{q_3} = \frac{c\varepsilon_1^q - s\varepsilon_2^q}{\varepsilon^q + \overline{\varepsilon}^q} \tag{27}$$

and $\varepsilon_\mu^q \equiv \partial_\mu \varepsilon^q$. The bar denotes the complex conjugation. Notice that, although Eq. (13) can be also written as Eq. (24), it is not covariant under an arbitrary gauge transformation. The reason is that now M_μ^q must have the special form defined as in Eq. (26). But under a transformation T_s the result $\widetilde{M}_\mu^q = T_s(M_\mu^q)$ probably is not as such. For this reason using a method similar to Chinea (1981) we consider the following differential equation

$$\widetilde{M}_\mu^q S^q = S^q M_\mu^q + \partial_\mu S^q \tag{28}$$

where $\widetilde{M}_\mu^q$ is still defined by Eq. (26). However, ε^q must be substituted by a solution $\tilde{\varepsilon}^q$. And the matrix $S^q \in SU_q(\eta(J))$. Its element $(S^q)^i_j = (S^q)^i_j\,(\varepsilon^q, \overline{\varepsilon}^q, \tilde{\varepsilon}^q, \overline{\tilde{\varepsilon}}^q; x, y; q)$ $(i, j = 1, 2, 3)$ is a function of those variables. Although Eq. (28) is more complex, the concrete solution can be carried out as in Chinea (1981). For two solutions S_1^q and S_2^q we define an operator "$\circ$" as

$$\left(S_1^q \circ S_2^q\right)(\varepsilon^q, \overline{\varepsilon}^q, \tilde{\varepsilon}^q, \overline{\tilde{\varepsilon}}^q) = S_1^q(\varepsilon^q, \overline{\varepsilon}^q, \tilde{\varepsilon}^q, \overline{\tilde{\varepsilon}}^q)\, S_2^q(\varepsilon^q, \overline{\varepsilon}^q, \tilde{\varepsilon}^q, \overline{\tilde{\varepsilon}}^q) \tag{29}$$

It can be easily proved that by this operator the set G^q of all solutions of S^q's form a group, which can be regarded as a subgroup of $SU(\eta(J))$. In fact, it is related to a general q-deformation of the Geroch group (Zhong, 1992).

Therefore, we obtain an explanation of the q-deformed gravitational fields. Let φ^q be a fundamental vector field installed in the space-time. The gauge group is taken as G^q. And the covariant derivative and the gauge field corresponding to the connection M_μ^q in Eq. (26), respectively, are denoted by ∇_μ^q and $F_{\mu\nu}^q$. Therefore, a q-deformed gravitational fields solution is just a ε^q, which makes $F_{\mu\nu}^q = 0$, i.e., the "interaction" vanishes. In other words, ε^q makes the equation

$$\nabla_\mu^q \varphi^q = 0 \tag{30}$$

integral, if φ^q exits. The above discussion also holds for the case of $q = 1$. Now a solution ε is to make the gauge field null, i.e.

$$\begin{aligned}
F_{\mu\nu}^q &= \partial_\mu M_\nu - \partial_\nu M_\mu - [M_\mu, M_\nu] = 0,\\
M_\mu &= m_\mu^1 J_3 + m_\mu^2 J_+ + m_\mu^3 J_-,\\
m_\mu^i &= m_\mu^{q_i}(\varepsilon^q \to \varepsilon)
\end{aligned} \tag{31}$$

Notice that, the transformations T_s and T_s^q, in essence, are the Bäcklund transformations (Chinea, 1981) generating new solutions $\tilde{\varepsilon}$ and $\tilde{\varepsilon}^q$, respectively.

Now, we discuss the physical difference between field $F_{\mu\nu}^q$ and field $F_{\mu\nu}$, i.e., the problem of the q-deformation effects. M_μ^q can be written as

$$\begin{aligned}
M_\mu^q &= A_\mu^q + D_\mu^q,\\
A_\mu^q &= \sqrt{\frac{\Delta}{2}}\left(m_\mu^{q_1} J_3 + m_\mu^{q_2} J_+ + m_\mu^{q_3} J_-\right),\\
D_\mu^q &= \left(\frac{\Delta}{2} - \sqrt{\frac{\Delta}{2}}\right) m_\mu^{q_1} J_3 = \left(\sqrt{\frac{\Delta}{2}} - \frac{\Delta}{2}\right) \frac{\partial_\mu \psi^q}{f^q} J J_3
\end{aligned} \tag{32}$$

Since we require that the transformation rule about M_μ^q must continuously translate into the transformation rule about M_μ as $q \to 1$, the transformation rules about A_μ^q and D_μ^q, respectively, must be

$$A_\mu^q \to \tilde{A}_\mu^q = S^q A_\mu^q (S^q)^{-1} + (\partial_\mu S^q)(S^q)^{-1},$$

$$D^q_\mu \to \tilde{D}^q_\mu = S^q D^q_\mu (S^q)^{-1} \tag{33}$$

So A^q_μ still can be taken as a connection, and D^q_μ is a vector under the gauge transformation. This means that M^q_μ has been split into parts with distinct properties, and D^q_μ, in fact, corresponds to a "torsion." Obviously $D^q_\mu \to 0$ when $q \to 1$. Thus the deformation effect, in view of the fixed absolute coordinates (t, z, τ, σ), is to generate a torsion in the first place. Now $F^q_{\mu\nu}$ can be split into

$$F^q_{\mu,\nu} = \hat{F}^q_{\mu\nu} + T^q_{\mu\nu}, \tag{34}$$

$$\hat{F}^q_{\mu\nu} = \partial_\mu A^q_\nu - \partial_\nu A^q_\mu - \left[A^q_\mu, A^q_\nu\right], \tag{35}$$

$$T^q_{\mu\nu} = \partial_\mu D^q_\nu - \partial_\nu D^q_\mu - \left[A^q_\mu, D^q_\nu\right] + \left[A^q_\nu, D^q_\mu\right] \tag{36}$$

where the gauge field $\hat{F}^q_{\mu\nu}$ is the curvature part of the field, $T^q_{\mu\nu}$ is a combination of torsions, and clearly $T^q_{\mu\nu} = 0$ when $q = 1$.

To further clarify the physical meaning of the q-deformation, we take the covariant divergences in both sides of Eq. (35), and it can be written as

$$\sum_{\mu=1}^{2} \nabla^q_\mu F^q_{\mu\nu} = j^q_\nu,\ j^q_\nu = \left(\frac{\Delta}{2} - \sqrt{\frac{\Delta}{2}}\right) J \nabla^q_\mu \left[\frac{\partial_\mu f^q \partial_\nu \psi^q - \partial_\nu f^q \partial_\mu \psi^q}{(f^q)^2} J_3 \right.$$
$$\left. + \sqrt{\frac{\Delta}{2}} \frac{(\partial_\mu - \partial_\nu)\Psi^q}{f^q} \left(m^{q_2}_\nu - m^{q_2}_\mu\right) J_+ - \left(m^{q_3}_\nu - m^{q_3}_\mu\right) J_- \right] \tag{37}$$

It is easily seen from Eq. (37) that generally $j^q_\nu \neq 0$ unless $\psi^q = \text{const}$. According to Eq. (33), j^q_ν does not vanish under arbitrary gauge transformation. Thus j^q_ν can be regarded as a source. This indicates that when $\psi^q \neq \text{const.}$, the q-deformation effect is to split the null gauge field $F^q_{\mu\nu}$ into two parts with distinct properties, particularly the gauge field $\hat{F}^q_{\mu\nu}$ with a source appearance.

The above q-deformation effects, in essence, stem from the characteristic noncommutative relation of the quantum group $SU_q(\eta(J))$

$$q^{J^q_3} \cdot J^q_\pm = q^{\pm 1} J_\pm \cdot q^{J^q_3} \tag{38}$$

For the proper gravitational fields ($q = 1$), Eq. (38) changes into a commutative relation, and both D^q_μ and j^q_ν vanish. Therefore, the proper gravitational fields actually correspond to some source-free, torsionless, and null gauge fields. By using the gauge transformations, one can obtain various gravitational field solutions.

4. CONCLUSION

We have shown that a nonlinear system of q-deformed double complex Ernst equation is obtained. The physical effects of the q-deformation are to generate a torsion in a proper gravitational field, to separate the gauge field into two parts with distinct properties, and to form a gauge field with source. The results in this paper can be extended in other directions. For example, about the Lie algebra of a non-Abelian Lie group one can consider an affine transformation similar to Eq. (6). Then the gauge field will be q-deformed. And one can obtain some new types of nonlinear equations, etc.. Of course, the results do not necessarily relate to a quantum group and the gravitational fields.

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